

Curved Momentum Space and Relative Locality

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Abstract

I briefly discuss the construction of a theory of particles with curved momentum space and its consequence, the principle of relative locality.

Dedicated to Professor Jerzy Lukierski on his 75th birthday.

Today, almost one hundred years after general relativity was first formulated, we are accustomed to the notion of curved spacetime. We know that the nontrivial spacetime geometry is in one to one correspondence with the dynamical gravitational field and how to relate the abstract mathematical structures of Riemannian (and, more generally, Cartan) geometry with physical concepts like distance, acceleration, tidal forces etc. Moreover, we understand that the source of the gravitational field is the energy-momentum of matter (and of gravity itself). This is possible because the dimension of the product of Newton's constant and the energy density has the dimension of inverse length, establishing in this way the curvature scale ("matter tells geometry how to curve".)

Already at the dawn of differential geometry Carl Friedrich Gauss noticed that the necessary prerequisite for the existence of a nontrivial (curved) geometry is the presence of a scale:

"The assumption that the sum of the three angles [of a triangle] is smaller than 180° leads to a geometry which is quite different from our (Euclidean) geometry, but which is in itself completely consistent. I have satisfactorily constructed this geometry for myself [...], except for the determination of one constant, which cannot be ascertained a priori. [...] Hence I have sometimes in jest expressed the wish that euclidean geometry is not true. For then we would have an absolute a priori unit of measurement."¹

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¹As cited in [1].

The necessity of the presence of the scale is easy to understand. Indeed a nontrivial geometry requires nonlinear structures and those can be constructed only if there is a scale that makes it possible to construct nonlinear expressions from fundamental, dimensionfull basic variables.

General relativity tells us that spacetime is curved, but what about another spaces that are of relevance in physics? One can interpret the Gauss' dictum as a statement that if a scale of some physical quantity is present in a theory, one could expect that the geometry of the corresponding manifold must be nontrivial. Or putting it in other words: "everything is curved unless it cannot be."

There are several examples that support this claim. Special relativity introduces a scale of velocity, and one could suspect that the manifold of (three) velocities may possess nontrivial structures. And indeed it does! Contrary to Galilean mechanics in special relativity the velocity composition law is highly nontrivial

$$\vec{v} \oplus \vec{u} = \frac{1}{1 + \vec{u}\vec{v}/c^2} \left(\vec{v} + \frac{\vec{u}}{\gamma_v} + \frac{1}{c^2} \frac{\gamma_v}{1 + \gamma_v} (\vec{v}\vec{u})\vec{v} \right), \quad \gamma_v = \sqrt{1 - \vec{v}^2/c^2}. \quad (1)$$

This expression is neither symmetric nor associative and is related to deep mathematics [2]. It also has interesting physical consequences (Thomas precession).

The relativistic, four-momentum space is, arguably, even more important physically than the spacetime. Indeed virtually all physical measurements can be reduced to the measurements of energies and momenta of incoming particles of various kinds (probes) performed by measuring devices located at the origin of a coordinate system. It is only by observing the incoming probes that we can infer the properties of distance events [3], [4]. The question arises as to if we have good reasons to believe that the momentum space is an almost structureless Minkowski space, or it is conceivable perhaps that it could possess more intricate geometrical structures.

Following the Gauss' intuition a possible way of addressing this question is to look for a theory that could provide us with a momentum scale κ . Such a theory indeed exists, and is pretty well known [5].

In $2 + 1$ spacetime dimensions the Newton's constant G_3 has the dimension of inverse mass raising the hope that it may provide the sought momentum scale being a prerequisite for the emergence of a nontrivial momentum space geometry. This hope was fully confirmed by the dynamical model calculations [6], [7].

Let us briefly recall the picture that emerges from these papers. As it is well known, in $2 + 1$ dimensions gravity is described by a topological field theory, so that local degrees of freedom are not present. Consider a point massive particle coupled to gravity. Since the system *particle + gravity* has only a finite number of degrees of freedom one can solve it exactly to obtain an effective description of the particle that includes the back-influence of the gravitational field it creates. As it turns out [6] such effective theory can be described as a theory of the particle with curved momentum space, the curvature scale being G_3 , as expected. Similar conclusion has been reached in the case of a scalar

field coupled to quantum gravity in $2 + 1$ dimensions [8]: by integrating out in the path integral (Ponzano–Regge model) the gravitational degrees of freedom one obtains an effective action for the scalar field, which can be interpreted as an action for the field with curved momentum space.

And what about gravity in the physical $3 + 1$ dimensions? Now the Newton’s constant is the ratio of Planck length ℓ_P and the Planck mass M_P

$$\ell_P = \sqrt{\hbar G}, \quad M_P = \sqrt{\frac{\hbar}{G}}, \quad (2)$$

and therefore has the dimension of length over mass. However, one can imagine a regime, presumably of quantum gravity, in which the Planck length is negligible, while the Planck mass remains finite. This formally means that both \hbar and G go to zero, so that both quantum and local gravitational effects become negligible, while their ratio remains finite [9]. In more physical terms this regime is realized if the characteristic length scales relevant for the processes of interest are much larger than ℓ_P , while characteristic energies are comparable with the Planck energy. An example of such a process might be the gravitational scattering in the case when the longitudinal momenta are Planckian, while the transferred momentum is very small (as compared to M_P) [10], [11]. In the case of such processes we again encounter the situation that the momentum scale is present, and we expect to find a nontrivial geometry of the momentum space. Unfortunately, to date no specific model of this kind has been formulated.

In absence of concrete models derived directly from quantum gravity, let us just assume that momentum space has a nontrivial geometry, governed by the the momentum scale κ (presumably of order of the Planck mass) and let us try to derive the most general possible description of particles kinematics. In order to do so it is convenient to start with the discussion of the standard relativistic particles action, to see how could it be generalized.

The free relativistic particle action

$$S = - \int_{-\infty}^{+\infty} d\tau \, x^a \dot{p}_a + N (\eta^{ab} p_a p_b - m^2) \quad (3)$$

consists of two terms: the kinetic one $-x^a \dot{p}_a$, $a, b = 0, \dots, 3$ and the mass shell constraint $\eta^{ab} p_a p_b - m^2$ imposed by the Lagrange multiplier $N(\tau)$. It is worth noticing that the term $\eta^{ab} p_a p_b$ is nothing but the square of the Minkowski distance between the point \mathcal{P} in momentum space, with coordinates p_a and the momentum space origin \mathcal{O} with coordinates $p_a = 0$, calculated along the straight line joining these two points, i.e., the geodesic of the Minkowski space geometry.

The equations of motion resulting from this action are

$$\dot{p}_a = 0, \quad \eta^{ab} p_a p_b = m^2, \quad \dot{x}^a = N \eta^{ab} p_b. \quad (4)$$

It is also worth noticing that the action (3) is manifestly invariant under global Lorentz and local τ reparametrization symmetries, as well as under the

following global translations (up to the boundary terms)

$$\delta x^a = \xi^a, \quad \delta p_a = \delta N = 0. \quad (5)$$

Finally, it should be stressed that the action (3) can be meaningfully written down only if the coordinates x^a and p_a on both spacetime and momentum space are given *a priori*.

The model presented above was essentially the standard theory of free relativistic particles. Let us now extend this theory to describe a system of interacting particles. In order to do that we make use of the intuitions coming from the theory of Feynman diagrams. Let us describe the system with a single three-particles interaction in details – the generalization to the more complex cases is going to be obvious.

Let us assume that we have to do with some number of incoming particle and the outgoing ones, interacting in a vertex in which momentum is going to be conserved. First we must slightly modify the free particle action (3). By choosing appropriately the form of the affine parameter τ , one can restrict its range to be from $-\infty$ to 0 for the incoming particles and from 0 to ∞ for the outgoing ones. I will implicitly assume this choice and not write the range of integration in what follows. Thus the “bulk” free particles action reads

$$S_{free} = - \sum_{\mathcal{I}} \int d\tau \, x_{\mathcal{I}}^a \dot{p}_a^{\mathcal{I}} + N_{\mathcal{I}} (\eta^{ab} p_a^{\mathcal{I}} p_b^{\mathcal{I}} - m_{\mathcal{I}}^2) \quad (6)$$

In order to include interaction one has to add the momentum conservation at the vertex. This condition is imposed by the adding to the action (6) the following term

$$S_{int} = z^a \widetilde{\sum_{\mathcal{I}} p_a^{\mathcal{I}}}, \quad (7)$$

where z^a is the Lagrange multiplier imposing the momentum conservation and the tilde over the sum indicates that the incoming (four-) momenta are to be taken with the plus, while the outgoing with the minus, sign. The bulk equations of motion resulting from the action $S_{free} + S_{int}$ are just the standard free particle equations (4), for each particle. The equation of motion for the variable z^a is the momentum conservation at the vertex. The only remaining equation follows from the boundary contribution at 0 to the variation over $p_a^{\mathcal{I}}$ combined with variation of the S_{int} over this variable, which leads to the condition

$$x_{\mathcal{I}}^a(0) = z^a, \forall_{\mathcal{I}}. \quad (8)$$

The meaning of this last condition is clear: the worldlines of all the particles meet at the interaction point, in which the physical coordinates of the particles are all equal the “interaction coordinate” z^a . Therefore the interaction is local in spacetime, and all inertial observers (Lorentz transformed, translated) agree that the interaction is local. Locality is *absolute*.

Let us now investigate what kind of the implicit geometrical data concerning the flat momentum space have been used to write the interaction vertex (7).

Clearly, in order to do so we had to know how to add momenta and what is the negative momentum. In the case of momentum space being a linear space both these operations are easy to define: For the summation we take vectors from the origin to the points $\mathcal{P}^{\mathcal{I}}$ with coordinates $p_a^{\mathcal{I}}$ and simply sum them. The negative momentum can be constructed by taking such a vector, multiply it by -1 and find which point of the manifold the tip of the resulting vector points to. Clearly this is going to be the point $-\mathcal{P}^{\mathcal{I}}$ with coordinates $-p_a^{\mathcal{I}}$. It is worth stressing the obvious fact that the interaction vertex is invariant under permutation of incoming and/or outgoing particles, since the momentum conservation rule is abelian.

We will generalize now the flat momentum space particle mechanics to the case of curved momentum space, pointing out the differences between these two cases and the new effects that arises as a consequence of the momentum space curvature. In what follows I will restrict myself mainly to the case when the momentum space has the structure of a group manifold. This is motivated by simplicity of such a choice and also by the $2 + 1$ dimensional experience. More general cases can be also considered (see [3] and [12]).

Let us start with the generalization of the kinetic term in the action (6). Since we have to do with the curved momentum space manifold now, the coordinate of a point \mathcal{P} is now p_μ with “curved” index μ . Therefore we must use a tetrad of the curved momentum space metric $e_a{}^\mu$ (notice the upside-down position of indices, as compared with the curved spacetime case) and the natural generalization of the kinetic term reads

$$-x^a e_a{}^\mu(p) \dot{p}_\mu.$$

It should be stressed that since $e_a{}^\mu(p)$ is a nonlinear function of momenta one needs to have a scale of mass available to define it

$$e_a{}^\mu(p) = \delta_a^\mu + \frac{1}{M_P} C^{\mu\nu}{}_a p_\nu + \dots$$

Even in the case of a Lie group momentum manifold there are many metrics that one can use to form the tetrad $e_a{}^\mu$ (see e.g., [13] for clear discussion.) However in this case there is a natural construction of the kinetic term. To see how it works notice that on any Lie group there is a natural Lie algebra valued one form

$$\omega = g^{-1} dg, \quad g \in G.$$

Since this form is Lie algebra valued, we can always write it in some coordinates as $g^{-1} dg = e_a{}^\mu(p) dp_\mu \gamma^a$, where γ^a is some basis of the Lie algebra. Defining now the basis of the dual space by

$$\langle \sigma_a, \gamma^b \rangle = \delta_a^b,$$

we see that the kinetic term can be compactly rewritten as

$$- \langle x, g^{-1} \dot{g} \rangle \tag{9}$$

with $x \equiv x^a \sigma_a$ is an element of a space dual to the Lie algebra of the Lie group G .

Having discussed the kinetic term we can turn now to the mass shell condition which would generalize the standard $p^2 - m^2$ of special relativity. The generalization is rather obvious once one realizes that p^2 has a geometric meaning of a square of the flat momentum space distance between the point \emptyset with coordinates $p_a = 0$ (i.e., zero energy and linear momentum) and the point \mathcal{P} with coordinates p_μ . Therefore, in the curved momentum space case one could do exactly the same, defining the distance $D^2(p)$ which is calculated with the help of the metric $g_{\mu\nu} = \eta_{ab} e_a^\mu e_b^\nu$ (although, in principle one can use any other available metric on the group momentum manifold.) Putting all this together we see that in the case of the group momentum manifold one has to replace the action (6) with

$$S_{free} = - \sum_{\mathcal{I}} \int d\tau \langle x, g_{\mathcal{I}}^{-1} \dot{g}_{\mathcal{I}} \rangle + N_{\mathcal{I}} (D^2(p_{\mathcal{I}}) - m_{\mathcal{I}}^2) . \quad (10)$$

With the free particles action (10) at hands, one can now turn to generalizing the interaction term (7). Since in our case the momentum space is a group manifold, we can make use of the group structure to define the momentum composition rule. (In the more general case one has to introduce the composition structure. It is shown in [3] and [12] that this amounts to defining a connection on the momentum space. Such connection does not need to be, in general, a Levi-Civita connection of the metric of the tetrad in the kinetic term and/or the one implicitly present in the mass-shell condition).

With every point on the momentum group manifold \mathcal{P} we can associate a group element $g(\mathcal{P})$. Let us define a function $K : G \rightarrow R^4$ such that for the group element $g(\mathcal{P})$, $K_\mu(g(\mathcal{P}))$ are coordinates of the point \mathcal{P} . Suppose now that we have a vertex with two incoming momenta $p_\mu^{(1)}$ and $p_\mu^{(2)}$, and the outgoing one $p_\mu^{(3)}$. Making use of the group structure one defines the momentum conservation at the vertex by

$$g_{(1)}^{-1} g_{(2)}^{-1} g_{(3)} = 1 \text{ or } K_\mu \left(g_{(1)}^{-1} g_{(2)}^{-1} g_{(3)} \right) = 0 \quad (11)$$

(notice that with the incoming momenta I associated inverse group elements in agreement with the convention used in (7).) It is important to realize at this point that the group multiplication is not abelian in general and thus the ordering of group elements in (11) matters. In the case of trivalent vertex with two incoming and one outgoing particles there are two possible momentum conservation conditions

$$K_\mu \left(g_{(1)}^{-1} g_{(2)}^{-1} g_{(3)} \right) = 0 \text{ and } K_\mu \left(g_{(2)}^{-1} g_{(1)}^{-1} g_{(3)} \right) = 0 .$$

We will not dwell here into the discussion which of the two (and many more for higher valent vertices) has to be chosen and how this can be decided; we will just assume that each vertex comes with a priori decided ordering of incoming

and outgoing lines. Then the momentum conservation rule (7) is replaced by the analogous, nonlinear expression

$$z^\mu K_\mu \left(p^{(1)}, p^{(2)}, p^{(3)} \right) . \quad (12)$$

where

$$K_\mu \left(p^{(1)}, p^{(2)}, p^{(3)} \right) = K_\mu \left(g_{(1)}^{-1}(p^{(1)}) g_{(2)}^{-1}(p^{(2)}) g_{(3)}(p^{(3)}) \right) ,$$

if the first conservation rule above is used.

The bulk equations of motion following from the action (10) are (for each particle, so that we do not add the label \mathcal{I})

$$e_a^\mu \dot{p}_\mu = 0 \quad (13)$$

$$\frac{d}{d\tau} (e_a^\mu x^a) - \frac{\partial D^2}{\partial p_\mu} = 0 \quad (14)$$

$$D^2(p) - m^2 = 0 \quad (15)$$

where in (15) we make use of the reparametrization of τ invariance to set $N = 1$. Notice that it follows from these equations that the momentum is conserved along the worldlines, $\dot{p}_\mu = 0$, because the tetrad e_a^μ in eq. (13) is invertible by definition.

The equations (13)–(15) should be appended by equations resulting from the presence of the vertex. One of them comes from variation over z^μ and enforces the momentum conservation at the vertex

$$K_\mu \left(p^{(1)}, p^{(2)}, p^{(3)} \right) = 0 , \quad (16)$$

The second comes from the boundary contribution at $\tau = 0$ of the variations over momenta of the bulk particle actions (we will ignore a possible contribution of the boundaries at $\tau = \mp\infty$ for incoming/outgoing particles), to wit

$$\bar{x}_\mathcal{I}^a e_\mu^a(p^\mathcal{I}) = z^\nu \frac{\partial K_\nu}{\partial p_\mu^\mathcal{I}} , \quad \bar{x}_\mathcal{I}^a \equiv x_\mathcal{I}^a(0) . \quad (17)$$

This condition relates the coordinates in two spacetimes: the ambient spacetime with coordinates x^a and the spacetime of “interaction coordinates” z^μ which can be associated with the (co)-tangent space to the momentum manifold at the origin. It is important to notice that this relation depends in general on all the momenta carried by the particles whose worldlines meet at the vertex. This momentum dependence disappear only in the case in which K_μ is a linear function of momenta and $e_\mu^a = \delta_\mu^a$ i.e., if and only if the momentum space is flat.

Before turning into the discussion of physical interpretation and consequences of this result, let us return to the symmetries of the action (10), (12). we will discuss here only the translational symmetry (the delicate issue of Lorentz symmetry is addressed in [14].) The kinetic term of (10) is invariant, up to a boundary term, under the following global translational symmetry, with parameter ξ , generalizing (5). To see how it works let us first rewrite the kinetic term as

$$-x_\mathcal{I}^\mu \dot{p}_\mu^\mathcal{I} , \quad x_\mathcal{I}^\mu \equiv x_\mathcal{I}^a e_a^\mu(p^\mathcal{I}) .$$

The most general transformation that leaves the bulk action

$$S_{free} = - \sum_{\mathcal{I}} \int d\tau x_{\mathcal{I}}^{\mu} \dot{p}_{\mu}^{\mathcal{I}} + N_{\mathcal{I}} (D^2(p_{\mathcal{I}}) - m_{\mathcal{I}}^2)$$

invariant up to a boundary term reads

$$\delta x_{\mathcal{I}}^{\mu} = \xi^{\nu} \frac{\partial F_{\nu}(p)}{\partial p_{\mu}^{\mathcal{I}}}$$

The resulting boundary term in variation of the action is $-\xi^{\nu} \dot{F}_{\nu}$ and it can be canceled by the vertex term if $F_{\nu} = K_{\nu}$ and $\delta z^{\nu} = \xi^{\nu}$ and therefore the total action is invariant under the translations

$$\delta x_{\mathcal{I}}^{\mu} = \xi^{\nu} \frac{\partial K_{\nu}(p)}{\partial p_{\mu}^{\mathcal{I}}}, \quad \delta z^{\mu} = \xi^{\mu}, \quad \delta p = \delta N = 0. \quad (18)$$

Returning to the original variables we find

$$\delta x_{\mathcal{I}}^a = \xi^{\nu} \frac{\partial K_{\nu}(p)}{\partial p_{\mu}^{\mathcal{I}}} e_{\mu}^a(p^{\mathcal{I}}). \quad (19)$$

It should be noticed that an extension of this result to the case of many vertices is not straightforward, see [15] for the detailed discussion.

Equation (17) along with the equations (18) and (19) governing the translational invariance lead us directly to the issue of relative locality. As a result of the translational invariance (18) there exist an observer for whom the interaction coordinates of the process z^{μ} are zero. Then it follows from (17) that the particles coordinates $\bar{x}_{\mathcal{I}}^a$ for all the particles vanish as well. For this particular observer the process is *local*. However *locality is relative*: for any translated observer $\bar{x}_{\mathcal{I}}^a \neq \bar{x}_{\mathcal{J}}^a$ if $\mathcal{I} \neq \mathcal{J}$ the translations of the worldlines in the ambient space with coordinates $x_{\mathcal{I}}^a$ depend not only on the momentum carried by the particle \mathcal{I} but also on the momenta carried by all the particles interacting in the vertex. This is a striking novel feature of the curved momentum space as compared to special relativity where locality has an absolute meaning, as all the special relativistic observers agree on what is local and what is not. The principle of relative locality is discussed further in [3] and [4].

This completes this brief discussion of the theory of particles with curved momentum space. There are many open problems that must be still investigated. The most pressing one concerns the spacetime. As we showed above there are two spacetimes involved in the construction: the ambient spacetime with x^a coordinates and the interaction spacetime with coordinates z^{μ} , which coincide in special relativity and are approximately identical if the energies/momenta of particles involved in the process are very small as compared with the momentum scale M_P . The question arises as to which of them is physical i.e., in which the operatively well defined spacetime measurements (of distance, speed, etc) are taking place. This is an important question because it is likely that theories

with curved momentum space might be testable by this type of measurements (see [12] and [15].)

Another problem is to turn the qualitative argument leading to curved momentum space in four dimensions, presented at the beginning of this article to a solid derivation from the first principles. Work on this question is in progress.

Last but not least it would be of great interest to extend the construction from particles to fields. At this moment only some particular models of free scalar field with curved momentum space are known, and it is not known how to generalize them to interacting theories involving higher spin fields (especially spinors and Yang-Mills fields.) It would be very important to be able to construct at least the leading order corrections to the Standard Model lagrangian (as it has been done in the case of theories with Lorentz invariance violations.) If successful this research program will make it possible to investigate the consequences of curved momentum space and relative locality and make contact with elementary particles phenomenology.

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